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1981 J. Phys. A: Math. Gen. 14 1863

(http://iopscience.iop.org/0305-4470/14/8/013)

J. Phys. A: Math. Gen. 14 (1981) 1863–1885. Printed in Great Britain

A geometrical definition of spinors from 'orientations' in three-dimensional space leading to a linear spinor visualisation

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Received 8 December 1980, in final form 16 February 1981

Abstract. An analysis of relative orientations of objects ('orientation' in the sense that an object is brought from one orientation to another by means of some rotation) leads to a geometrical definition of spinors and to the relationship between spinors and vectors. The spinor notation obtained is in some respects more basic than the conventional, complex notation, which contains some ambiguity. It is shown that spinors can be described by additive, vector-like entities in the real three-dimensional space (R^3). These entities are linear in the spinor components and transform under rotations in a simple manner (different from that of vectors). Each ordinary, complex component of the spinor can be obtained by a projection on a complex plane embedded in R^3. Spinors—in all their aspects including the sign—acquire a clear and natural geometrical significance in this approach. Furthermore, it unifies in a geometrical manner the SU(2) transformations of spinors, the connection between spinors and vectors and the recent spinor visualisation model of Hellsten. The latter is also derived here in a simple manner.

1. Introduction

Spinors—the two-component complex quantities associated with spin-$\frac{1}{2}$ states—have often been regarded as abstract entities that are only indirectly connected with the geometry of the Euclidean three-dimensional real space (R^3). Since vectors can be expressed as quadratic functions of spinors, they have been used as the link between spinors and R^3, as, for example, in the spinor definition of Cartan (1966, p 41). This view gives no interpretation of the sign of the spinor, which is lost in the quadratic expression, and the connection with R^3 is only indirect. The linear nature of the spinors is hidden; for example, to start with the vectors associated with two spinors and form the vectors that correspond to the sum of the spinors is obviously complicated. This is even impossible to carry through unless the signs of the spinors are determined by some non-vectorial quantities. Other approaches, such as the description of spinor geometry by Veblen (1933) and the nullflag interpretation of spinors by Penrose (1968), are also of this kind.

Conventionally, the spinor space is a linear, complex two-space (C^2). Sometimes (e.g. Gel'fand *et al* 1963), the spinors are regarded merely as the quantities in C^2 that are transformed under rotations by matrices in SU(2)—the simplest double-valued representation of the rotation group. Then, the linear nature of the spinors is explicit by definition. However, the connection between the geometries of C^2 and R^3 remains unclear, and the transformation of the spinor components in C^2 under rotations in R^3 is quite involved.

Although it is then tempting to regard spinors merely as abstract entities—a view that certainly has had a great impact on the theory of quantum mechanics—this is not necessary and does not give a fair picture of the spinors. Recently, H Hellsten (1979) has given a new and concrete way of looking at spinors in R^3 that does not suffer from the deficiencies contained in the quadratic relation between spinors and vectors. He has also generalised it to the Minkowski space and included twistors (Hellsten 1980a, b). Hellsten's model in R^3 is treated in some detail in § 5 of this work.

In this paper we shall give a complementary picture of spinors in R^3. We shall obtain the notion of a spinor and the relationship between spinors and vectors from an analysis of relative orientations of objects. Here, the concept of orientation is used in the sense that an object can be brought from one arbitrary orientation to another by means of a rigid rotation or a reversal (rotation + inversion through the origin). The concept should become clear from what follows (the triad formulation, § 3.2, supplies a more strict definition). Note that our meaning of 'orientation' is different from that used for example in differential geometry. We shall assume that the objects used are non-symmetrical (e.g. a cube with distinctly marked faces or a triplet of distinguishable vectors which do not lie in one plane, 'orientable objects') so that unique orientations can be defined from them.

Consider a collection of equal, superimposable objects which have a fixed point (origin) in common, such that all objects can be brought to coincide by means of different rotations (for simplicity we restrict ourselves to rotations here, but what is said below is true for reversals as well). In the following we shall make use of a slight abstraction. Let us assign an orientation to each object (they are distinct for non-coinciding objects) and talk about transformations of orientations instead of change in orientations of the objects under rotations. Thus two distinct orientations are related by some rotation, that would bring the first one to the second if the latter were kept fixed. The rotation is uniquely defined up to a rotation through $2\pi\eta$ (η some integer), i.e. it specifies a unique axis of rotation and an angle of rotation (modulo 2π) measured in some definite sense around the axis. The perpendicular plane that contains the origin will be called the plane of rotation. We shall conventionally label a rotation by the parameters $\hat{n}$ (unit normal to the plane of rotation) and φ (angle of rotation measured in the positive sense around $\hat{n}$).

From a physical point of view, a common rotation of *all* objects has no meaning, and only relative transformations are meaningful. Thus we need some reference objects which are exempt from the transformation (interpreted in the active sense), while everything else is changed. As regards orientations, we take some reference orientation (Ω_r) to which all orientations are referred. A change in Ω_r equals a passive rotation which leaves the other orientations unaffected. However, the *relation* (some rotation) between a certain, arbitrary orientation (Ω) and Ω_r becomes changed. Obviously, the same change in the relation occurs during the corresponding active transformation when all orientations except Ω_r are changed in the reverse manner. The active view of the transformations will be held in the following unless otherwise stated, and the values of the rotation parameters will normally be chosen in the active sense.

From the relation between an orientation (Ω) and the reference orientation (Ω_r), we shall define the concept of a spinor associated with Ω, in § 2. This association will express the invariant nature of the spinor itself (cf vectors under passive rotations), while its components, which are obtained from the relation, depend on the reference orientation chosen. In § 3 we shall see how the relation between two orientations, none of which equals Ω_r, defines a vector, which fact gives rise to a direct link between spinors and vectors. Furthermore, the relationship between spinors and triads (there defined) is derived, and the effect of spatial inversions is investigated. Section 4 is devoted to spinor visualisations, and finally in § 5 the connection with the spinor visualisation model of Hellsten (1979) is given.

2. The relation between orientations and spinors

Let us first choose an arbitrary, but fixed, reference orientation Ω_r in R^3 (it can be thought to be associated in some way with the frame of reference chosen). Consider some orientation Ω that is related to Ω_r by a rotation (inversions will be considered in § 3.4). Let the parameters $(\hat{n}, -\varphi)$ define a rotation that would bring Ω to coincide with Ω_r. For simplicity—but somewhat inconsequently—we shall in the following say that the rotation $(\hat{n}, \varphi)$ would rotate Ω_r to Ω. If we perform an active rotation through ψ around the axis given by $\hat{m}$, Ω is transformed to, say, Ω' while Ω_r is fixed. The new orientation Ω' is related to Ω_r by means of a new rotation $(\hat{n}', \varphi')$, which would rotate Ω_r to Ω'. Obviously, $(\hat{n}', \varphi')$ defines the rotation that is the combination of the rotation $(\hat{n}, \varphi)$ followed by $(\hat{m}, \psi)$:

$$\begin{array}{ccc} & \Omega & \\ {\scriptstyle R(\hat{n},\varphi)}\nearrow & & \searrow{\scriptstyle R(\hat{m},\psi)} \\ \Omega_r & \xrightarrow{R(\hat{n}',\varphi')} & \Omega'. \end{array} \qquad (2.1)$$

The value of $(\hat{n}', \varphi')$ is uniquely determined by $(\hat{n}, \varphi)$ if its change is followed continuously when ψ is varied from 0 to the final value (this will always be required in the following).

The combination of rotations is easily performed by means of a spherical triangle as shown in figure 1. The corners of the triangle lie on the rotation axes, while the angles equal half the rotation angles. This is a direct consequence of the fact that any rotation can be performed by means of consecutive reflections in two planes that intersect at the rotation axis and form an angle of half the rotation angle to each other. The construction of a combination of such reflections—and hence of rotations—is shown geometrically by Misner *et al* (1973, pp 1137–9) (cf the figure caption of our figure 1).

From figure 1 we see that the resultant rotation axis n' moves in a plane (Λ) when ψ is varied ($\hat{m}$ is kept fixed), i.e. $\hat{n}'$ describes a circle of unit radius in Λ. Note that Λ is determined by $(\hat{n}, \varphi)$ and $\hat{m}$, $\Lambda = \Lambda(\hat{m}; \hat{n}, \varphi)$; it is the $(\hat{n}, \hat{m})$ plane rotated through $-\varphi/2$ around $\hat{n}$ (the case that $\hat{m}$ and $\hat{n}$ are parallel is trivial).

This construction gives the transformation $(\hat{n}, \varphi) \rightarrow (\hat{n}', \varphi')$ under the active rotation $(\hat{m}, \psi)$. It is easy to see that a rotation through 2π $(\psi = 2\pi)$ gives the result $\hat{n}' = -\hat{n}$ since $\psi/2 = \pi$ is used, and a further rotation through 2π is necessary in order to come back to the original $\hat{n}$. The circular motion of $\hat{n}'$ under the rotation is not simple, since the angular velocity of $\hat{n}'$ is non-uniform when ψ is increased at an even rate (unless $\hat{m}$ is perpendicular to the plane of the circle). However, it is possible to modify $\hat{n}'$ slightly such that its motion can be decomposed into simpler motions, as we shall see soon.

Figure 2 gives a different view of figure 1 with the $(\hat{n}, \hat{m})$ and $(\hat{n}', \hat{m})$ planes and the tangent vectors $\bar{h}$, $\bar{k}$, $\bar{h}'$, $\bar{k}'$, $\bar{f}$ and $\bar{f}'$ of the spherical triangle shown. The latter two

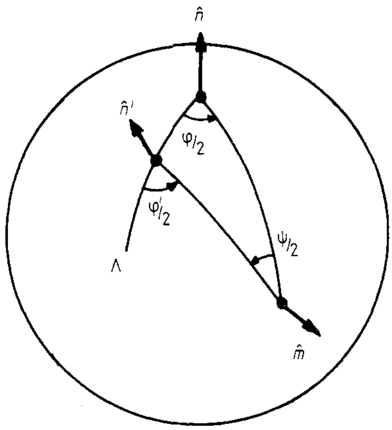

Figure 1. Composition of rotations illustrated by a spherical triangle. Its sides are great circles of the sphere, and $\hat{n}$, $\hat{m}$ and $\hat{n}'$ lie on the intersections between the planes of the circles. The rotation $(\hat{n}', \varphi')$ is the result of the rotation $(\hat{n}, \varphi)$ followed by $(\hat{m}, \psi)$. It can also be seen as the result of consecutive reflections (see text) in the planes $(\hat{n}\,\hat{n}')$ and $(\hat{n}, \hat{m})$ (i.e. rotation $(\hat{n}, \varphi)$) followed by $(\hat{n}, \hat{m})$ and $(\hat{n}', \hat{m})$ (i.e. rotation $(\hat{m}, \psi)$). Since the middle two reflections cancel, the resultant reflection planes are $(\hat{n}, \hat{n}')$ and $(\hat{n}', \hat{m})$ (i.e. rotation $(\hat{n}', \varphi')$).

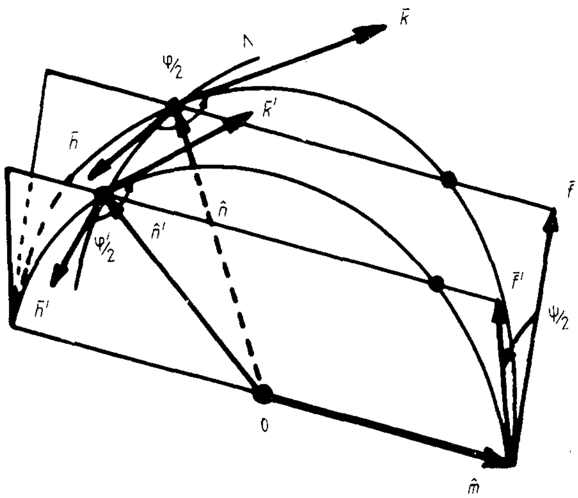

Figure 2. A detailed view of the illustration in figure 1. The tangent vectors of the spherical triangle are shown. $\bar{h}$ and $\bar{h}'$ lie in the $(\hat{n}, \hat{n}')$ plane (Λ), while $\bar{k}$ and $\bar{f}$ $(\bar{k}'$ and $\bar{f}')$ lie in the plane $(\hat{n}, \hat{m})$ $((\hat{n}', \hat{m}))$. The angles between the tangent vectors are also shown (cf figure 1).

vectors are chosen such that they constitute the projections of $\hat{n}$ and $\hat{n}'$ on a plane perpendicular to $\hat{m}$. We take

$$\begin{aligned} \bar{f} &= \hat{n} - (\hat{n}\cdot\hat{m})_m, & \bar{f}' &= \hat{n}' - (\hat{n}'\cdot\hat{m})\hat{m}, \\ \bar{h} &= \hat{n}' - (\hat{n}\cdot\hat{n}')\hat{n}, & \bar{h}' &= -[\hat{n} - (\hat{n}'\cdot\hat{n})\hat{n}'], \\ \bar{k} &= \hat{m} - (\hat{n}\cdot\hat{m})\hat{n}, & \bar{k}' &= \hat{m} - (\hat{n}'\cdot\hat{m})\hat{n}'. \end{aligned} \tag{2.2}$$

It is easy to see that

$$|\bar{f}| = |\bar{k}|, \qquad |\bar{f}'| = |\bar{k}'|, \qquad |\bar{h}| = |\bar{h}'|. \tag{2.3}$$

Evidently (figure 2)

$$\bar{h} \times \bar{k} = |\bar{h}||\bar{k}| \sin(\tfrac{1}{2}\varphi)\hat{n} = |\bar{h}||\bar{f}| \sin(\tfrac{1}{2}\varphi)\hat{n}. \tag{2.4}$$

From equation (2.2) it follows that $(\bar{h} \times \bar{k}) \cdot \hat{n} = (\bar{h}' \times \bar{k}') \cdot \hat{n}'$. Inserting equation (2.4) and the corresponding primed relation, we obtain

$$|\bar{f}| \sin \tfrac{1}{2}\varphi = |\bar{f}'| \sin\tfrac{1}{2}\varphi'. \tag{2.5}$$

This motivates us to introduce

$$\begin{aligned} \bar{N} &= \hat{n} \sin \tfrac{1}{2}\varphi, & \bar{F} &= \bar{f} \sin \tfrac{1}{2}\varphi, \\ \bar{N}' &= \hat{n}' \sin \tfrac{1}{2}\varphi', & \bar{F}' &= \bar{f}' \sin \tfrac{1}{2}\varphi', \end{aligned} \tag{2.6}$$

since the projections of $\bar{N}$ and $\bar{N}'$ on a plane perpendicular to $\hat{m}$ equal $\bar{F}$ and $\bar{F}'$ respectively, which have the same length. Furthermore, the angle between $\bar{F}'$ and $\bar{F}$ equals $\psi/2$ (cf $\bar{f}'$ and $\bar{f}$ in figure 2). The transformation $\bar{N} \to \bar{N}'$ under the active rotation $(\hat{m}, \psi)$ accordingly becomes a simple rotation through $\psi/2$ when projected on that plane. Since $\bar{N}'$ and $\hat{n}'$ are parallel (or antiparallel), the transformation of $\bar{N}'$ also takes place in the plane $\Lambda(\hat{m}; \hat{n}, \varphi)$. Obviously, $\bar{N}'$ moves in an elliptical orbit when ψ is varied (see figure 3) and its motion is simple to follow from the circular, uniform motion of the projection $\bar{F}'$.

Define $\bar{E}$ from

$$\bar{E} = \hat{m} \times \bar{F} = \hat{m} \times \bar{N}. \tag{2.7}$$

Since $|\bar{E}| = |\bar{F}|$ we obviously have (figure 3)

$$\bar{F}' = \cos(\tfrac{1}{2}\psi)\bar{F} + \sin(\tfrac{1}{2}\psi)\bar{E}. \tag{2.8}$$

Define $\bar{G}$ in the plane Λ from the requirement that its projection equals $\bar{E}$ (figure 3).

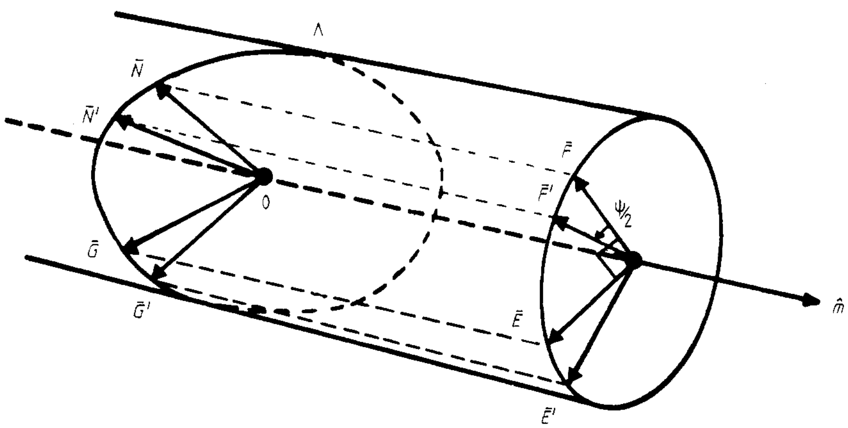

Figure 3. The orbit of the 'vector' $\bar{N}'$ ($\bar{N}$ transformed under the rotation $(\hat{m}, \psi)$) is obtained as the intersection between a cylindrical surface centred at $\hat{m}$ and the tilted plane Λ. The projections on a plane perpendicular to $\hat{m}$ are shown to the right. $\bar{F}$ and $\bar{F}'$ are orthogonal to $\bar{E}$ and $\bar{E}'$ respectively, which are the the projections of $\bar{G}$ and $\bar{G}'$.

From the condition that $\bar{N}'$ moves in the $(\bar{N}, \bar{G})$ plane and from equation (2.8) it follows that

$$\bar{N}' = \cos(\tfrac{1}{2}\psi)\bar{N} + \sin(\tfrac{1}{2}\psi)\bar{G}. \tag{2.9}$$

Analogously, $\bar{G}'$ is defined in Λ from $\bar{E}' = \hat{m} \times \bar{N}'$. It is simple to see that $\bar{G}'$ is obtained from $\bar{N}'$ by increasing $\psi/2$ with $\pi/2$, i.e. equation (2.9) yields

$$\bar{G}' = -\sin(\tfrac{1}{2}\psi)\bar{N} + \cos(\tfrac{1}{2}\psi)\bar{G}. \tag{2.10}$$

Equations (2.9)–(2.10) can be used to investigate the motion of $\bar{N}'$ projected on $\hat{m}$ (while equation (2.8) gave the perpendicular components of the motion). Let us define ν and ν' to be the projections of $\bar{G}$ and $\bar{G}'$ respectively on $\hat{m}$:

$$\nu = \bar{G}\cdot\hat{m}, \qquad \nu' = \bar{G}'\cdot\hat{m}. \tag{2.11}$$

Equations (2.9)–(2.10) give

$$\bar{N}'\cdot\hat{m} = \cos(\tfrac{1}{2}\psi)\bar{N}\cdot\hat{m} + \sin(\tfrac{1}{2}\psi)\nu, \qquad \nu' = -\sin(\tfrac{1}{2}\psi)\bar{N}\cdot\hat{m} + \cos(\tfrac{1}{2}\psi)\nu. \tag{2.12}$$

Thus, the two-vector $\bar{D}'$ that has the components $(\nu', \bar{N}'\cdot\hat{m})$ performs a circular motion in an imagined plane, where the initial $\bar{D}$ is defined by $(\nu, \bar{N}\cdot\hat{m})$. To illustrate this in R^3 we can let $\hat{m}$ define one axis of the plane (i.e. $\bar{D}'\cdot\hat{m} = \bar{N}'\cdot\hat{m}$) and choose the ν axis in any direction perpendicular to $\hat{m}$. The situation is depicted in figure 4.

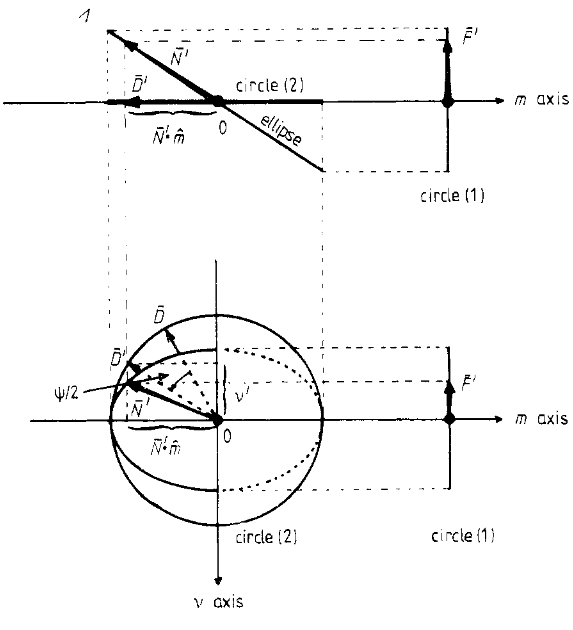

Figure 4. The transformation of $\bar{N}'$ and ν' under the rotation $(\hat{m}, \psi)$. A side and a top view of the elliptic orbit of $\bar{N}'$ are shown. The right circle (1) is a projection of the ellipse. The projections of $\bar{N}:\bar{F}'$ (see figure 3) and $\bar{N}'\cdot\hat{m}$ are also shown. The latter quantity is also the projection of $\bar{D}'$ on the m axis. The projection of D' on the perpendicular axis equals ν'. The two-vector $\bar{D}'$ moves with a rotation angle $\psi/2$ in the circular orbit (2) shown. The plane of this circle (2) is seen from the side in the upper figure (i.e. the ν axis lies in Λ, the plane of the ellipse) but can also be rotated together with $\bar{D}$ and $\bar{D}'$ around $\hat{m}$ to an arbitrary angle.

Since ν plays a crucial role in equations (2.12), the proper quantity to investigate further is $(\bar{N}, \nu)$ which contains four components. In the Appendix it is shown that $\nu = \cos(\varphi/2)$, and hence we have

$$(\bar{N}, \nu) = (\hat{n} \sin(\tfrac{1}{2}\varphi), \cos(\tfrac{1}{2}\varphi)). \tag{2.13}$$

We have seen (equations (2.8) and (2.12)) how its transformation under active rotations $(\hat{m}, \psi)$ can be decomposed by projection into two simple rotations through $\psi/2$, one $(\bar{F}')$ around $\hat{m}$ and one $(\bar{D}')$ in an arbitrary plane containing $\hat{m}$. Thus, the transformation rule is determined entirely by $\hat{m}$ and ψ, and is the *same* for all $(\bar{N}, \nu)$. It also follows that it is *independent of the reference orientation.* Furthermore, the sum of two 'vectors' (the quotation marks indicate that they do not transform like ordinary vectors) that rotate through $\psi/2$ also rotates through $\psi/2$, so the transformation is *linear.* Note that the transformation of an ordinary vector and that of the 'vector' $\bar{N}$ differ in only two ways. The angles of rotation around $\hat{m}$ are ψ and $\psi/2$ respectively, and the projection of a vector on $\hat{m}$ is constant while that of $\bar{N}$ is the projection of another circular motion.

The quantity $(\bar{N}, \nu)$ is, by definition, a *spinor* (we shall see below that this definition is equivalent to the conventional one, but cf also § 3.4). Its transformation formula is contained in equations (2.9) and (2.12). From the definition of $\bar{G}$ and from equations (2.7) and (2.11) it follows that

$$\bar{G} = \hat{m} \times \bar{N} + \nu \hat{m} \tag{2.14}$$

and hence we obtain

$$\bar{N}' = \cos(\tfrac{1}{2}\psi)\bar{N} + \sin(\tfrac{1}{2}\psi)\hat{m} \times \bar{N} + \sin(\tfrac{1}{2}\psi)\nu\hat{m}, \qquad \nu' = \cos(\tfrac{1}{2}\psi)\nu - \sin(\tfrac{1}{2}\psi)(\bar{N} \cdot \hat{m}). \tag{2.15}$$

Note that $\bar{N}^2 + \nu^2 = (\bar{N}')^2 + (\nu')^2 = 1$. It is simple to check that equation (2.15) remains valid when $\hat{m}$ is parallel to $\hat{n}$. Then the ellipse degenerates to a line along $\hat{n}$.

To see the relationship to the conventional way of writing a spinor, let us express $\bar{N}$ in its components $\bar{N} = \begin{pmatrix} N_x \\ N_y \\ N_z \end{pmatrix}$ and introduce the compact notation

$$u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} = \begin{pmatrix} N_z + \mathrm{i}\nu \\ N_x + \mathrm{i}N_y \end{pmatrix}. \tag{2.16}$$

Note that $\langle u | u \rangle = \bar{N}^2 + \nu^2$. In this notation equation (2.15) can be written

$$u' = \boldsymbol{B}_{\hat{m}}(\psi) u \tag{2.17}$$

where

$$\boldsymbol{B}_{\hat{m}}(\psi) = \cos(\tfrac{1}{2}\psi)\mathbf{1} + \sin(\tfrac{1}{2}\psi)\hat{m} \cdot \bar{\boldsymbol{b}}, \qquad \hat{m} \cdot \bar{\boldsymbol{b}} = m_x \boldsymbol{b}_x + m_y \boldsymbol{b}_y + m_z \boldsymbol{b}_z,$$

$$\boldsymbol{b}_x = \boldsymbol{i} = \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \boldsymbol{b}_z = \boldsymbol{k} = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix},$$

$$\boldsymbol{b}_y = \boldsymbol{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$\bar{\boldsymbol{\sigma}} = \mathrm{i}\bar{\boldsymbol{b}}$ are the conventional Pauli matrices, and equation (2.17) is the common spinor transformation formula (Cartan 1966, pp 46–7 Misner *et al* 1973, pp 1137, 1148). **1**, ***i***,

$\boldsymbol{j}$ and $\boldsymbol{k}$ satisfy

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -\mathbf{1} \tag{2.18}$$

and are accordingly quaternion basis elements, a well known fact.

The spinor transformation can be written somewhat differently if we introduce (using equation (2.13))

$$\boldsymbol{B}(\bar{N}, \nu) = \boldsymbol{B}_{\hat{n}}(\varphi) = \nu\mathbf{1} + \bar{N}\cdot\bar{\boldsymbol{b}} \tag{2.19}$$

i.e. the spinor written as a quaternion. Equation (2.17)—as well as equation (2.15)—is equivalent to

$$\boldsymbol{B}(\bar{N}', \nu') = \boldsymbol{B}_{\hat{m}}(\psi)\boldsymbol{B}(\bar{N}, \nu). \tag{2.20}$$

This expresses the composition of rotations (2.1)—which was our point of departure—by means of spinor transformation matrices. Equation (2.13) merely introduces the well known Euler–Olinde–Rodrigues parameters of the rotation. The relation between the C^2 and the quaternion notations (equations (2.16) and (2.19)) of the spinor is

$$u = \boldsymbol{B}(\bar{N}, \nu)\begin{pmatrix} \mathrm{i} \\ 0 \end{pmatrix}. \tag{2.21}$$

So far, only normed spinors have been considered. Obviously, a general spinor is obtained by multiplying $(\bar{N}, \nu)$ by a positive scale factor. Since this generalisation is very simple to perform, we shall for simplicity assume that the spinors are normed in the following unless otherwise stated.

As will be evident at several places in what follows, the four-component notation $(\bar{N}, \nu)$ of a spinor is in many ways more basic than the complex one $\begin{pmatrix} u_+ \\ u_- \end{pmatrix}$. In fact, the complex notation is not necessary in order to write a spinor as a quaternion and to obtain equations (2.17)–(2.20). If the quaternion basis elements above are substituted by the 4×4 real matrices (cf the complex $\bar{\boldsymbol{b}}$-matrices)

$$\begin{pmatrix} \mathbf{0} & -\boldsymbol{j} \\ -\boldsymbol{j} & \mathbf{0} \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \qquad \begin{pmatrix} -\boldsymbol{j} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{j} \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix},$$

where $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\boldsymbol{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and if we use the real column matrix $\begin{pmatrix} N_z \\ \nu \\ N_x \\ N_y \end{pmatrix}$ instead of u, equations (2.17)–(2.20) remain valid. It is easy to check that equation (2.17) is then identical to equation (2.15) written in matrix form.

To end this section, we shall make a comment on the sign indeterminicity of the spinors. The rotation $(\hat{n}, \varphi)$ that relates the orientation Ω and the reference orientation Ω_r is uniquely determined up to a rotation angle of $2\pi\eta$ (η any integer). From equation (2.13) we see that this leads to two possible signs of the spinor associated with Ω. Furthermore, from equation (2.15) it follows that an active rotation through 2π around $\hat{m}$ changes the sign of $(\bar{N}, \nu)$, and also of $\hat{n}$, as we saw earlier from figure 1 (the last statement is true unless $\hat{m}$ is parallel to $\hat{n}$, the only case where $\sin(\varphi/2)$ changes sign and where $\hat{n}$ is constant). However, this rotation transforms the plane of rotation associated with $(\hat{n}, \varphi)$ into itself, only its normal has a reversed sign.

The rotation $(\hat{n}, \varphi)$ would bring a point P to another point Q on the unit circle in that plane. Let us now start again and take any circle with two arbitrary points P and Q specified. It defines *two* rotations with $0 \leqslant \varphi \leqslant 2\pi$ that would bring P to Q along the circle (the rotation axis is perpendicular to the plane of the circle). The rotations differ in sign of $\hat{n}$ (i.e. in rotation direction) and define the spinors $\pm(\bar{N}, \nu)$. Thus the initial sign indeterminicity of the spinor can be expressed as the geometrical property of R^3: *there exist two unit normals defining the very same plane.* By choosing one normal of the plane (i.e. a sign of the normal), and hence one sign of the spinor, we have removed the indeterminicity, since the choice can be followed by continuity during rotations.

3. Relations between spinors and vectors

3.1. *Orientations, spinors and vectors*

Let us regard two distinct orientations Ω_1 and Ω_2 related in the usual manner to the reference orientation Ω_r by the rotations $(\hat{n}_1, \varphi_1)$ and $(\hat{n}_2, \varphi_2)$ respectively. From equation (2.13) we obtain the spinors $(\bar{N}_1, \nu_1)$ and $(\bar{N}_2, \nu_2)$ which are associated with Ω_1 and Ω_2 (each associated spinor is unique up to a sign). The two orientations also determine a rotation $(\hat{l}, \chi)$ (χ determined up to $2\pi\eta$, η any integer) that would bring Ω_1 to Ω_2 if the latter were kept fixed. This rotation also relates the two spinors (provided the multiple of 2π is chosen properly)

$$\boldsymbol{B}(\bar{N}_2, \nu_2) = \boldsymbol{B}_{\hat{l}}(\chi)\boldsymbol{B}(\bar{N}_1, \nu_1) \tag{3.1}$$

(cf equation 2.20)).

Let us now, for simplicity, regard the effect of a *passive* rotation $(\hat{m}, \psi)$. It rotates Ω_r, while Ω_1 and Ω_2 are invariant. Thus the rotation axis $(\hat{l})$ and angle (χ) are also unchanged, which means that $\hat{l}$ transforms like a vector and χ like a scalar under rotations (passive or active). Under the active rotation $(\hat{m}, \psi)$, the direction of $\hat{l}$ is transformed to that of, say, $\hat{l}'$. The transformation of the vector components $\hat{l} = \begin{pmatrix} l_x \\ l_y \\ l_z \end{pmatrix}$ can be obtained from equation (3.1), which we write as

$$\boldsymbol{B}(\bar{N}_2, \nu_2)\boldsymbol{B}^{-1}(\bar{N}_1, \nu_1) = \boldsymbol{B}_{\hat{l}}(\chi) = \cos(\tfrac{1}{2}\chi)\mathbf{1} + \sin(\tfrac{1}{2}\chi)\hat{l} \cdot \bar{\boldsymbol{b}}. \tag{3.2}$$

Using the spinor transformation (2.20), we obtain

$$\boldsymbol{B}_{\hat{l}'}(\chi) = \boldsymbol{B}_{\hat{m}}(\psi)\boldsymbol{B}_{\hat{l}}(\chi)\boldsymbol{B}^{\dagger}_{\hat{m}}(\psi) \tag{3.3}$$

where the facts that χ is a scalar and $\boldsymbol{B}^{-1} = \boldsymbol{B}^{\dagger}$ are utilised. Thus we have for any vector $\bar{v}$

$$\bar{v}' \cdot \bar{\boldsymbol{b}} = \boldsymbol{B}_{\hat{m}}(\psi)(\bar{v} \cdot \bar{\boldsymbol{b}})\boldsymbol{B}^{\dagger}_{\hat{m}}(\psi) \tag{3.4}$$

which is the usual transformation formula of a vector written as a quaternion (Cartan 1966, pp 45–6).

The vector and the scalar are related by equation (3.2) to the spinors. Using equations (2.18)–(2.19), we obtain from the coefficients of $\bar{\boldsymbol{b}}$ and $\mathbf{1}$ in equation (3.2) (cf equation (2.15))

$$\hat{l} \sin \tfrac{1}{2}\chi = \bar{N}_1 \times \bar{N}_2 + \nu_1 \bar{N}_2 - \nu_2 \bar{N}_1, \qquad \cos \tfrac{1}{2}\chi = \bar{N}_1 \cdot \bar{N}_2 + \nu_1\nu_2. \tag{3.5}$$

Hence we may define a *scalar multiplication*

$$(\bar{N}_1, \nu_1)\cdot(\bar{N}_2, \nu_2) = \bar{N}_1\cdot\bar{N}_2 + \nu_1\nu_2 \tag{3.6}$$

and a *vector multiplication*

$$(\bar{N}_1, \nu_1)\times(\bar{N}_2, \nu_2) = \bar{N}_1\times\bar{N}_2 + \nu_1\bar{N}_2 - \nu_2\bar{N}_1 \tag{3.7}$$

for spinors. Let us introduce the notation $U = (\bar{N}, \nu)$, and turn to general (not normed) spinors. We accordingly have the spinor norm

$$\|U\| = (U\cdot U)^{1/2} \tag{3.8}$$

which is equal to the usual one, and the scalar and vector multiplications

$$U_1\cdot U_2 = \|U_1\|\|U_2\|\cos(\tfrac{1}{2}\chi), \qquad U_1\times U_2 = \|U_1\|\|U_2\|\sin(\tfrac{1}{2}\chi)\hat{l}. \tag{3.9}$$

The geometrical interpretation of these formulae is simple as regards the connection between the orientations Ω_1 and Ω_2 associated with the normed spinors and $(\hat{l}, \chi)$ (it follows from the construction above). Note that the same $\hat{l}$ is determined from several pairs of orientations. The relation between $(\hat{n}_1, \varphi_1)$, $(\hat{n}_2, \varphi_2)$ and $(\hat{l}, \chi)$ is given geometrically by a spherical triangle as in figure 1, which follows from a comparison of equations (3.1) and (2.20).

For spinors written in the usual form (2.16) as complex two-component entities, another—but related—form of the relation between spinors and vectors is commonly used. The matrix $u_1u_2^\dagger$ can always be decomposed,

$$u_1u_2^\dagger = V\mathbf{1} + \mathrm{i}\bar{v}\cdot\bar{\boldsymbol{b}}, \tag{3.10}$$

where in general $\bar{v}$ and V are complex. Under rotations, $\bar{v}$ obviously transforms like a vector (equations (3.4)) and V like a scalar. A short calculation gives

$$V^2 = v_x^2 + v_y^2 + v_z^2 = (\bar{v})^2. \tag{3.11}$$

If equation (2.16) is inserted in equation (3.10), a straightforward computation gives the real and imaginary parts

$$\mathrm{Re}\ V = \tfrac{1}{2}(\bar{N}_1, \nu_1)\cdot(\bar{N}_2, \nu_2), \qquad \mathrm{Im}\ \bar{v} = \tfrac{1}{2}(\bar{N}_1, \nu_1)\times(\bar{N}_2, \nu_2), \tag{3.12a}$$

$$\mathrm{Im}\ V = \tfrac{1}{2}(\bar{N}_1, \nu_1)\cdot(\bar{N}'_2, \nu'_2), \qquad \mathrm{Re}\ \bar{v} = -\tfrac{1}{2}(\bar{N}_1, \nu_1)\times(\bar{N}'_2, \nu'_2), \tag{3.12b}$$

where (for spinor two) $N'_x = -N_y$, $N'_y = N_x$, $N'_z = -\nu$ and $\nu' = N_z$. Written in complex form (equation (2.16)), this means that $u'_2 = \mathrm{i}u_2$. The geometrical interpretation of this will be discussed later (see equation (3.22)). Because of these relations it is reasonable to think of equation (3.9) as more basic than equation (3.10).

3.2. *The triad formulation*

The relation between spinors, vectors and orientations can be formulated in a powerful, unified manner by use of *triads*. We define a triad to be a triplet of orthogonal vectors of equal length and take this length as its norm. A triad is normed if the common length is unity. As a starting point we introduce the transformation operator $\boldsymbol{A}_{\hat{n}}(\varphi)$ for vector rotations

$$\bar{v}' = \boldsymbol{A}_{\hat{n}}(\varphi)\bar{v}. \tag{3.13}$$

When the rotation is active, it relates the different directions of $\bar{v}$ and $\bar{v}'$ respectively in space. If we write the vector components as column vectors $\bar{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$, $\boldsymbol{A}_{\hat{n}}(\varphi)$ is generally represented by the matrix

$$\boldsymbol{A}_{\hat{n}}(\varphi) = e^{\varphi \hat{n} \cdot \bar{\boldsymbol{a}}} = \boldsymbol{1} + \sin \varphi (\hat{n} \cdot \bar{\boldsymbol{a}}) + (1 - \cos \varphi)(\hat{n} \cdot \bar{\boldsymbol{a}})^2 \tag{3.14}$$

where

$$\boldsymbol{a}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{a}_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{a}_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let us now define the unit vectors $\hat{\xi}$, $\hat{\eta}$ and $\hat{\zeta}$ from the basis vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$ according to

$$\hat{\xi} = \boldsymbol{A}_{\hat{n}}(\varphi)\hat{x}, \qquad \hat{\eta} = \boldsymbol{A}_{\hat{n}}(\varphi)\hat{y}, \qquad \hat{\zeta} = \boldsymbol{A}_{\hat{n}}(\varphi)\hat{z}. \tag{3.15}$$

Together, they form a normed triad $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$ that is rotated through φ around $\hat{n}$ relative to the frame of reference $(\hat{x}, \hat{y}, \hat{z})$. Written as column vectors, $\hat{\xi}$, $\hat{\eta}$ and $\hat{\zeta}$ obviously equal the columns of the transformation matrix, $\boldsymbol{A}_{\hat{n}}(\varphi) = (\hat{\xi}, \hat{\eta}, \hat{\zeta})$.

Introducing the Euler–Olinde–Rodrigues parameters (equation (2.13)) in equation (3.14) and using $\bar{N}^2 + \nu^2 = 1$, we can write

$$\boldsymbol{A}(\bar{N}, \nu) = \boldsymbol{A}_{\hat{n}}(\varphi) = (\bar{N}^2 + \nu^2)\boldsymbol{1} + 2\nu(\bar{N} \cdot \bar{\boldsymbol{a}}) + 2(\bar{N} \cdot \bar{\boldsymbol{a}})^2. \tag{3.16}$$

Thus the components of $\boldsymbol{A}$ are homogeneous quadratic functions of $(\bar{N}, \nu)$. Note that the orientation of the triad $(\hat{x}, \hat{y}, \hat{z})$ serves as a reference orientation, and that $(\bar{N}, \nu)$ according to the definition in § 2 is a spinor associated with the orientation of $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$. In this manner we have obtained an explicit expression

$$(\hat{\xi}, \hat{\eta}, \hat{\zeta}) = (\bar{N}^2 + \nu^2)\boldsymbol{1} + 2\nu(\bar{N} \cdot \bar{\boldsymbol{a}}) + 2(\bar{N} \cdot \bar{\boldsymbol{a}})^2 \tag{3.17}$$

of the relation between spinors and orientations.

Let us now perform the rotation $(\hat{m}, \psi)$. The triad is transformed:

$$(\hat{\xi}', \hat{\eta}', \hat{\zeta}') = \boldsymbol{A}_{\hat{m}}(\psi)(\hat{\xi}, \hat{\eta}, \hat{\zeta}). \tag{3.18}$$

The orientation of the rotated triad is related to that of the frame of reference by means of a rotation $(\hat{n}', \varphi')$:

$$(\hat{\xi}', \hat{\eta}', \hat{\zeta}') = \boldsymbol{A}_{\hat{n}'}(\varphi')(\hat{x}, \hat{y}, \hat{z}) = \boldsymbol{A}(\bar{N}', \nu')\boldsymbol{1}.$$

Thus, equation (3.18) equals

$$\boldsymbol{A}(\bar{N}', \nu') = \boldsymbol{A}_{\hat{m}}(\psi)\boldsymbol{A}(\bar{N}, \nu) \tag{3.19}$$

which is the analogue of equation (2.20). As before, we require continuity, i.e. $\lim_{\psi \to 0}(\boldsymbol{N}', \nu') = (\bar{N}, \nu)$. From equation (3.19) one can again derive equation (2.15) (the proof is rather lengthy).

According to equation (3.17), a normed spinor and a normed triad are associated with each other. General spinors are obtained by multiplying the normed spinors by positive scale factors. The triad $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ associated with a particular spinor $(\bar{N}, \nu)$ is generally taken to be

$$(\bar{\xi}, \bar{\eta}, \bar{\zeta}) = (\bar{N}^2 + \nu^2)\mathbf{1} + 2\nu(\bar{N}\cdot\bar{\boldsymbol{a}}) + 2(\bar{N}\cdot\bar{\boldsymbol{a}})^2.$$

It differs from $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$ by the square of the scale factor. It is evident what is meant by equal orientations of two triads with different norm, and hence one may also talk about the orientation of $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ relative to that of $(\hat{x}, \hat{y}, \hat{z})$. This defines the spinor up to the scale factor, which is easily determined from the norm of the triad.

Obviously, relations between spinors and vectors are obtained by picking out any one of the vectors in each triad. As we shall see, one can also obtain these relations from equation (3.10) if the spinor $U = (\bar{N}, \nu)$ is written in the complex form (2.16). This gives the connection between the triad formulation and the traditional one.

If we take $u_1 = u_2 = u$ in equation (3.10) (for simplicity, u is assumed to be normed), we obtain a real vector that we call $\bar{v}^{\mathrm{I}}$:

$$uu^{\dagger} = V^{\mathrm{I}}\mathbf{1} + \mathrm{i}\bar{v}^{\mathrm{I}}\cdot\bar{\boldsymbol{b}}. \tag{3.20}$$

Since $\bar{v}^{\mathrm{I}}$ is real, the scalar V^{I} equals its length (equation (3.11)). A direct calculation gives

$$\bar{v}^{\mathrm{I}} = \tfrac{1}{2}\begin{pmatrix} A_{xz} \\ A_{yz} \\ A_{zz} \end{pmatrix} = \tfrac{1}{2}\mathbf{A}(\bar{N}, \nu)\hat{z} = \tfrac{1}{2}\hat{\zeta}. \tag{3.21}$$

The result can be interpreted in terms of equation (3.12). From the definition of the vector product, it is obvious that $\mathrm{Im}\,\bar{v}^{\mathrm{I}} = 0$. Using equation (3.9), we also find from equations (3.21) and (3.12*b*) that $U \times U' = -\hat{\zeta} = \sin(\chi/2)\hat{l}$ where $(\hat{l}, \chi)$ defines a rotation that transforms U to U' (cf equations (3.1) and (3.9)). Thus we can take $\hat{l} = \hat{\zeta}$ and $\chi = -\pi$ (the other choices are equivalent). We already know that in complex notation $u' = \mathrm{i}u$, and we obtain

$$u' = \boldsymbol{B}_{\hat{\zeta}}(-\pi)u = \mathrm{i}u. \tag{3.22}$$

The primed spinor $(\bar{N}'_2, \nu'_2)$ in equation (3.12*b*) is accordingly obtained from $(\bar{N}_2, \nu_2)$ by the rotation $(\hat{\zeta}, -\pi)$, where $\hat{\zeta}$ is defined by $(\bar{N}_2, \nu_2)$, ie by $(\hat{n}_2, \varphi_2)$ inserted in equation (3.15). The general relation (3.22) will be proved later (equation (3.30)).

The components of u can be written $u_+ = \rho_+ \exp(\mathrm{i}\gamma_+)$ and $u_- = \rho_- \exp(\mathrm{i}\gamma_-)$ where ρ_+, ρ_- are real and positive and γ_+, γ_- are real. Then the components of $\bar{v}^{\mathrm{I}}$ obtained from equation (3.20) are

$$v_x^{\mathrm{I}} = \rho_+\rho_- \cos(\gamma_- - \gamma_+), \qquad v_y^{\mathrm{I}} = \rho_+\rho_- \sin(\gamma_- - \gamma_+), \qquad v_z^{\mathrm{I}} = \tfrac{1}{2}(\rho_+^2 - \rho_-^2). \tag{3.23}$$

This form will be useful later.

As the next step, we introduce the conjugate spinor (u^{c}) belonging to u:

$$u^{\mathrm{c}} = \boldsymbol{C}u^* \tag{3.24}$$

where

$$\boldsymbol{C} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It transforms like an ordinary spinor under rotations (this follows from equation (2.17) and $\boldsymbol{CB}^* = \boldsymbol{BC}$).

If we put $u_1 = u$ and $u_2 = u^c$ in equation (3.10), we obtain a complex, isotropic vector $\bar{v}^{\mathrm{II}}$:

$$u(u^c)^\dagger = V^{\mathrm{II}}\mathbf{1} + \mathrm{i}\bar{v}^{\mathrm{II}} \cdot \bar{\boldsymbol{b}}. \tag{3.25}$$

That the vector is isotropic, i.e. $(v_x^{\mathrm{II}})^2 + (v_y^{\mathrm{II}})^2 + (v_z^{\mathrm{II}})^2 = (V^{\mathrm{II}})^2 = 0$, is equivalent to the statement that its real and imaginary parts constitute two orthogonal vectors of equal length. A straightforward calculation shows that

$$\bar{v}^{\mathrm{II}} = \tfrac{1}{2}\begin{pmatrix} A_{xx} + \mathrm{i}A_{xy} \\ A_{yx} + \mathrm{i}A_{yy} \\ A_{zx} + \mathrm{i}A_{zy} \end{pmatrix} = \tfrac{1}{2}\boldsymbol{A}(\bar{N}, \nu)(\hat{x} + \mathrm{i}\hat{y}) = \tfrac{1}{2}(\hat{\xi} + \mathrm{i}\hat{\eta}). \tag{3.26}$$

An analysis of this result in terms of equation (3.12) (cf the derivation of equation (3.22)) gives the meaning of the c-conjugation:

$$u^c = \boldsymbol{B}_{\hat{\eta}}(\pi)u \tag{3.27}$$

where $\hat{\eta}$ is defined by the spinor (equation (3.15)). The standard components of $\bar{v}^{\mathrm{II}}$ are

$$v_{+1}^{\mathrm{II}} = \frac{1}{\sqrt{2}}(u_+)^2, \qquad v_0^{\mathrm{II}} = u_+u_-, \qquad v_{-1}^{\mathrm{II}} = \frac{1}{\sqrt{2}}(u_-)^2. \tag{3.28}$$

In his definition of spinors in three-dimensional space, Cartan (1966, p 41) starts from an isotropic vector that equals $(\hat{\xi} + \mathrm{i}\hat{\eta}) = -2v^{\mathrm{II}}$ and defines, in principle, a spinor from the square root of its +1 and −1 components. To prove that the spinors so defined transform linearly under rotations is obviously equivalent to proving in a direct manner that the transformation (3.19) is linear in $(\bar{N}, \nu)$. Since $\hat{\xi}$ and $\hat{\eta}$ are sufficient to specify the *right-handed* triad $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$, a definition of a spinor according to equation (2.16) is equivalent to Cartan's definition (unless general triads are considered, see § 3.4).

3.3. Spinor phases and directions, spin

In our definition of a normed spinor in real form $(\bar{N}, \nu)$, we made use only of orientations, and no direction in space was preferred. In complex form (equation (2.16)), on the other hand, the spinor defines an associated direction provided its phase factor is ignored, i.e. all spinors that differ only in phase define a common direction in space. This well known fact can be seen from equation (3.20), where the substitution $u \to \exp(\mathrm{i}\gamma)u$ does not change $\bar{v}^{\mathrm{I}} = \frac{1}{2}\hat{\zeta}$. Conversely, $\bar{v}^{\mathrm{I}}$ defines ρ_+, ρ_- and $\gamma_- - \gamma_+$ by equation (3.23), so all spinors with a common $\hat{\zeta}$ direction of the associated triad differ only in phase. That $\hat{\zeta}$ happens to be the associated direction of the spinor depends on the definition of the phase factor, i.e. on the definition (2.16) of u from $(\bar{N}, \nu)$. Another choice of complex notation for the spinor $(\bar{N}, \nu)$ may give a different associated direction.

The phase change under a rotation around $\hat{\zeta}$ can be derived from equation (3.3) by substituting $\hat{m} \to \hat{n}$, $\psi \to \varphi$, $\hat{l} \to \hat{z}$ and hence $\hat{l}' \to \hat{z}' = \hat{\zeta}$:

$$\boldsymbol{B}_{\hat{\zeta}}(\chi)\boldsymbol{B}(\bar{N}, \nu) = \boldsymbol{B}(\bar{N}, \nu)\boldsymbol{B}_{\hat{z}}(\chi). \tag{3.29}$$

Together with equations (2.17) and (2.21) this gives

$$\boldsymbol{B}_{\hat{\zeta}}(\chi)u = \mathrm{e}^{-\mathrm{i}\chi/2}u \tag{3.30}$$

(note that $\hat{\zeta}$ is defined by u, cf equation (3.15) with $(\hat{n}, \varphi)$ of the spinor). If the phase of the spinor changes in time with the phase velocity ω, the orientation associated with the spinor will accordingly rotate around $\hat{\zeta}$ with the angular velocity 2ω (e.g. $\hat{\xi}$ and $\hat{\eta}$ of the triad will rotate likewise around $\hat{\zeta}$, cf Penrose (1968)). The direction of $\hat{\zeta}$ is the spin-$\frac{1}{2}$ eigendirection of the spinor used in quantum mechanics. This follows from equation (3.20), where $\bar{v}^{\mathrm{I}} = \frac{1}{2}\hat{\zeta}$ and $V^{\mathrm{I}} = \frac{1}{2}$:

$$\boldsymbol{S}_{\zeta} \equiv \tfrac{1}{2}\hat{\zeta}\cdot\bar{\boldsymbol{\sigma}} = uu^{+} - \tfrac{1}{2}\mathbf{1} \tag{3.31}$$

which gives $\boldsymbol{S}_{\zeta}u = \frac{1}{2}u$ ($\hbar = 1$).

In the general case we can subdivide the set of all orientations (normed spinors) into equivalence classes, each of which defines a distinct direction (unit vector) in the following manner. Take an arbitrary orientation (Ω_0) and associate a likewise arbitrary direction ($\hat{l}$) with it. By performing all possible rotations on Ω_0 and $\hat{l}$ simultaneously, all orientations are obtained and with each of them (Ω_0') a unique direction ($\hat{l}'$) is associated. All orientations with a common associated direction (e.g. $\hat{l}'$) form an equivalence class (L') (note that $\hat{l}'$ and $-\hat{l}'$ have different classes). The orientations in the class L belonging to the original $\hat{l}$ are obtained from Ω_0 by rotations around $\hat{l}$. Obviously, a rotation that takes $\hat{l}$ into $\hat{l}'$ makes a transformation $L \rightarrow L'$ (Ω_0 and Ω_1 in L are transformed into, say, Ω_0' and Ω_1' respectively in L'). If Ω_1 is obtained from Ω_0 by the rotation ($\hat{l}$, χ_1), then we obtain Ω_1' by rotating Ω_0' through χ_1 around $\hat{l}'$ (cf the derivation of equation (3.3)).

The corresponding equivalence classes of spinors are easily obtained from the relation between spinors and orientations. Two distinct spinors ($\bar{N}_1$, ν_1) and ($\bar{N}_2$, ν_2) in the same class are related to the vector $\hat{l}$ of the class by equation (3.5). The angle of rotation, χ can be used to define a 'phase angle' within each class (in our previous example we had Ω_0 given by ($\hat{\xi}$, $\hat{\eta}$, $\hat{\zeta}$) and $\hat{l} = \hat{\zeta}$). A spinning, orientable (classical) object is obviously described by such an equivalence class—where $\hat{l}$ is the spin axis—in a natural manner. (Note that $\hat{l}$ is a pseudo-vector in this case, cf the end of § 3.4)

3.4. Spatial inversions of spinors

We shall also say a few words about spatial inversions and reversals. By definition, we use equation (3.17) as the link between spinors and triads also in this case. Let ($\bar{N}'$, ν') denote the spinor after inversion. Then

$$(-\hat{\xi}, -\hat{\eta}, -\hat{\zeta}) = [(\bar{N}')^2 + (\nu')^2]\mathbf{1} + 2\nu'(\bar{N}'\cdot\bar{\boldsymbol{a}}) + 2(N'\cdot\bar{\boldsymbol{a}})^2. \tag{3.32}$$

Eliminating $\hat{\xi}$, $\hat{\eta}$ and $\hat{\zeta}$ from equations (3.17) and (3.32), we obtain a system of quadratic equations in ($\bar{N}$, ν) and ($\bar{N}'$, ν'). The only solutions to this system are

$$(\bar{N}', \nu') = \pm\mathrm{i}(\bar{N}, \nu). \tag{3.33}$$

Both the case with the plus sign and that with the minus sign occur. By convention we choose plus for ordinary spinors (spinors of the first kind, cf Cartan (1966)). Since ($\bar{N}$, ν) are real quantities, the spinors distinguish between right-handed and left-handed triads. Writing the spinor as a quaternion (equation (2.19)), $\nu\mathbf{1} + N_x\boldsymbol{i} + N_y\boldsymbol{j} + N_z\boldsymbol{k}$, we have to introduce biquaternions to describe inversions. Interpreted as a transformation (see § 2), the normed spinor $\mathrm{i}(\bar{N}, \nu)$ defines an inversion combined with a rotation, i.e. a reversal, which would transform the reference orientation (right-handed) to the orientation associated with the spinor (left-handed).

In contrast to this, the information about left- and right-handedness is lost in the conventional, complex notation (2.16) of a spinor. After an inversion ($\boldsymbol{I}$) we obtain

$$u' = \boldsymbol{I}u = \mathrm{i}u = \begin{pmatrix} \mathrm{i}N_z - \nu \\ \mathrm{i}N_x - N_y \end{pmatrix}. \tag{3.34}$$

This spinor can also be (formally) obtained from u by a rotation through $-\pi$ around $\hat{\zeta}$ (equation (3.30)). Thus a complex-valued spinor would correspond *both to one right-handed and one left-handed orientation* (the sign of $\hat{\zeta}$ is undefined). However, $\bar{v}^{\mathrm{I}}$ in equation (3.20) is well defined. It is a pseudovector and differs in sign from $\frac{1}{2}\hat{\zeta}$ after inversion. Note that $2\bar{v}^{\mathrm{I}}$ is the general spin eigendirection in equation (3.31).

Evidently, the spinor transformation matrix for a reversal, which is composed of the rotation $(\hat{m}, \psi)$ and an inversion, equals $\mathrm{i}\boldsymbol{B}_{\hat{m}}(\psi)$. For a reflection the rotation is taken through π around the normal $(\hat{m})$ of the plane of reflection (cf reflection of vectors), and the transformation matrix becomes (cf Cartan 1966, p 46)

$$\boldsymbol{M}_{\hat{m}} = \mathrm{i}\boldsymbol{B}_{\hat{m}}(\pi) = \mathrm{i}\hat{m}\cdot\bar{\boldsymbol{b}}. \tag{3.35}$$

We shall also consider inversions of conjugate spinors, which are of the second kind (Cartan 1966, p 50), i.e. transform with a minus sign in equation (3.33). This follows from equation (3.24) and the definition $\boldsymbol{I}(u^{\mathrm{c}}) = (\boldsymbol{I}u)^{\mathrm{c}}$. Equation (3.27) gives some further insight into this behaviour. It yields

$$\boldsymbol{I}(u^{\mathrm{c}}) = (\boldsymbol{I}u)^{\mathrm{c}} = \boldsymbol{B}_{-\hat{\eta}}(\pi)\mathrm{i}u = -\mathrm{i}u^{\mathrm{c}}.$$

Note the sign change of $\hat{\eta}$ due to the inversion of the associated triad.

Cartan (1966, p 49) has adopted a conjugate spinor $\tilde{u} = \mathrm{i}\boldsymbol{C}u^*$, that differs by a factor i from ours. It can be interpreted in two different ways due to the ambiguity in the complex notation of a spinor discussed above. The factor i may arise from an inversion (equation (3.34))

$$\tilde{u} = \mathrm{i}\boldsymbol{B}_{\hat{\eta}}(\pi)u = \boldsymbol{M}_{\hat{\eta}}u$$

where equation (3.35) has been used, or it may arise from a rotation (equation (3.30))

$$\tilde{u} = \mathrm{i}\boldsymbol{B}_{\hat{\eta}}(\pi)u = \boldsymbol{B}_{\hat{\eta}}(\pi)\boldsymbol{B}_{\hat{\zeta}}(-\pi)u = \boldsymbol{B}_{\hat{\xi}}(-\pi)u.$$

In the first case, conjugation equals a reflection in the $(\hat{\xi}, \hat{\zeta})$ plane, and the conjugated spinor has the associated triad $(\hat{\xi}, -\hat{\eta}, \hat{\zeta})$. In the second case the triad is $(\hat{\xi}, -\hat{\eta}, -\hat{\zeta})$. Note that we have $(\bar{v}^{\mathrm{II}})^{\sim} = \hat{\xi} - \mathrm{i}\hat{\eta} = (\bar{v}^{\mathrm{II}})^*$ (equation (3.26)) in both cases, which equality in fact constitutes the starting point of Cartan.

The definitions of scalar and vector multiplications (equations (3.6)–(3.7)) of spinors have to be completed when inversions are considered. From their derivation (equation (3.1)) it follows that it is arbitrary whether an inversion leaves the signs of $\hat{l}$ and χ unchanged or gives rise to a sign change in both. In the first case $\hat{l}$ is a pseudovector and χ a scalar, and one of the spinors in equations (3.5)–(3.7) has to be complex conjugated. In the second case $\hat{l}$ is a vector and χ is a pseudoscalar, and equations (3.5)–(3.7) can be used as they stand.

4. Spinor visualisations

Our construction of the spinor $(\bar{N}, \nu)$ in § 2 constitutes one possible way of visualising a spinor and its transformation under a rotation $(\hat{m}, \psi)$. The 'vector' $\bar{N}'$ moves in an elliptic orbit (figure 3) in the plane Λ, which is obtained by rotating the $(\hat{m}, \bar{N})$ plane through $|-\varphi/2|$ around $\bar{N}$ (or $-\varphi/2$ around $\hat{n}$, $-2\pi \leqslant \varphi \leqslant 2\pi$). The normal of Λ around which $\bar{N}'$ is orbiting in the positive direction is (see figures 3 and 5)

$$\bar{\lambda} = \bar{N} \times \bar{G}. \tag{4.1}$$

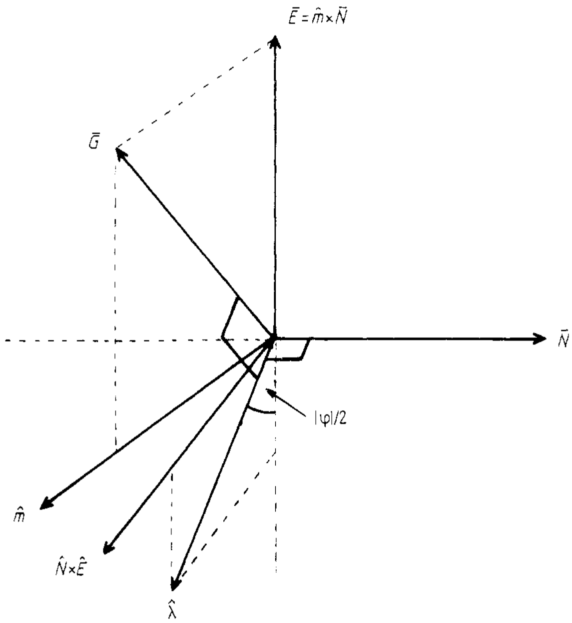

Figure 5. Illustration of the normal $\hat{\lambda}$ of Λ (the plane of motion of $\bar{N}'$), which equals the $(\bar{G}, \bar{N})$ plane (see equations (4.1)–(4.2) and figure 3).

Inserting equation (2.14) and $\bar{E} = \hat{m} \times \bar{N}$ (equation (2.7)), we obtain (for a normed spinor)

$$\hat{\lambda} = -\nu\hat{E} + |\bar{N}|(\hat{N} \times \hat{E}) \tag{4.2}$$

i.e the unit normal is defined by the component ν along $-\hat{E}$ and $|\bar{N}|$ orthogonally to the $(\bar{N}, \bar{E})$ plane on the same side as $\hat{m}$.

A disadvantage with this way of visualising a spinor is that the fourth component ν is treated separately. Instead, we shall give a better alternative, which we call the double-'vector' visualisation since it utilises two $\bar{N}$-'vectors'. It retains the symmetry shown by the spinor as a two-component complex quantity, and gives a simple geometric relation between the $\bar{N}$-'vectors' in R^3 and the spinor in C^2.

Let us interpret the (x, y) plane as a complex plane C ($\hat{x} = 1$, $\hat{y} = \mathrm{i}$). Then equation (2.16) shows that the projection of $\bar{N}$ on this plane equals $u_- = N_x + \mathrm{i}N_y$. Therefore, we shall give $(\bar{N}, \nu)$ used so far the index $-$. The relation between $(\bar{N}^-, \nu^-)$ and u can be expressed (equation (2.21)) by

$$u = \boldsymbol{B}(\bar{N}^-, \nu^-)\begin{pmatrix} \mathrm{i} \\ 0 \end{pmatrix}. \tag{4.3}$$

In order to obtain $u_+ = \alpha_+ + \mathrm{i}\beta_+$ geometrically in as simple a manner as $u_- = \alpha_- + \mathrm{i}\beta_-$, we define $(\bar{N}^+, \nu^+)$ from

$$u = \boldsymbol{B}(\bar{N}^+, \nu^+)\begin{pmatrix}0\\ \mathrm{i}\end{pmatrix} \tag{4.4}$$

where u is the same spinor as in equation (4.3). This gives $(\bar{N}^+, \nu^+) = (\alpha_+, -\beta_+, -\alpha_-, \beta_-)$. The projection of $\bar{N}^+$ on the (x, y) plane accordingly is $\alpha_+ - \mathrm{i}\beta_+ = u_+^*$ (as we shall see later, the complex conjugate is natural here). The quantities $(\bar{N}^+, \nu^+)$ and $(\bar{N}^-, \nu^-)$ are related through

$$N_x^+ = N_z^-, \qquad N_y^+ = -\nu^-, \qquad N_z^+ = -N_x^-, \qquad \nu^+ = N_y^-. \tag{4.5}$$

The geometric meaning of $(\bar{N}^+, \nu^+)$ can be obtained as follows. From equations (2.16) and (3.17) we see that $\begin{pmatrix}\mathrm{i}\\ 0\end{pmatrix}$ is associated with $(\hat{x}, \hat{y}, \hat{z})$ and $\begin{pmatrix}0\\ \mathrm{i}\end{pmatrix}$ with $(-\hat{x}, \hat{y}, -\hat{z})$. As before, u is associated with $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$, and equation (4.3) expresses that $(\bar{N}^-, \nu^-)$ defines a rotation that would bring $(\hat{x}, \hat{y}, \hat{z})$ to $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$. Analogously, equation (4.4) shows that $(\bar{N}^+, \nu^+)$ gives a rotation that would bring $(-\hat{x}, \hat{y}, -\hat{z})$ to $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$.

Obviously, both $(\bar{N}^+, \nu^+)$ and $(\bar{N}^-, \nu^-)$ transform according to equation (2.15). The transformations can be illustrated as in figure 3. The 'vectors' move in two different planes Λ^+ and Λ^-, the normals of which can be obtained from equation (4.2).

If we project $\bar{N}^+$ and $\bar{N}^-$ on the (x, z) plane we find the relation (from equation (4.5))

$$N_x^+ = N_z^-, \qquad N_z^+ = -N_x^-. \tag{4.6}$$

Thus, the projection of $\bar{N}^+$ is obtained from that of $\bar{N}^-$ by a 'rotation' through $\pi/2$ around $\hat{y}$ (see figure 6). This relation reduces the initial six degrees of freedom for two

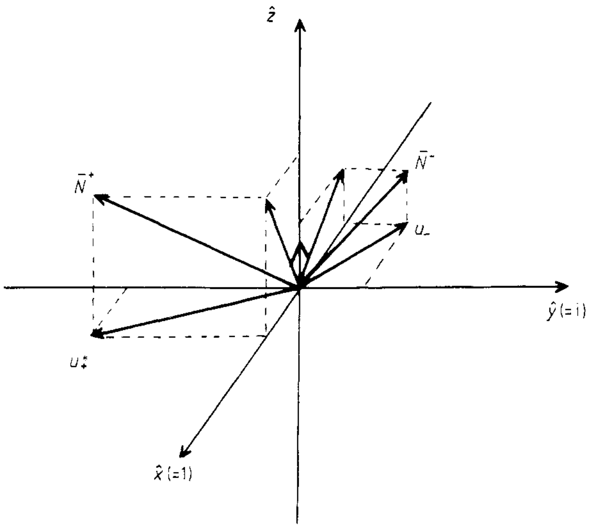

Figure 6. The relation between the double-'vector' visualisation $\begin{pmatrix}\bar{N}^+\\ \bar{N}^-\end{pmatrix}$ and the complex components $\begin{pmatrix}u_+\\ u_-\end{pmatrix}$ of a spinor. The projections of $\bar{N}^+$ and $\bar{N}^-$ on the (x, z) plane are always orthogonal and have equal length. Under rotational transformations the $\bar{N}$ 'vectors' move in planar, elliptic orbits (cf figure 3).

'vectors' to four, i.e. the same number as that of a general spinor (the common scale factor, which changes the lengths of $\bar{N}^+$ and $\bar{N}^-$ and the norm of the spinor, is the fourth degree of freedom).

The pair $\begin{pmatrix}\bar{N}^+\\ \bar{N}^-\end{pmatrix}$ subject to the condition (4.6) will constitute our double-'vector' visualisation of a spinor (figure 6). We summarise its salient properties.

(i) The complex components of u are obtained from the projections of $\bar{N}^+$ and $\bar{N}^-$ on the (x, y) plane.
(ii) $\bar{N}^+$ and $\bar{N}^-$ have direct and visually clear relations to orientations of triads.
(iii) The additivity of spinors corresponds to additivity of $\begin{pmatrix}\bar{N}^+\\ \bar{N}^-\end{pmatrix}$, where common vector addition (parallelogram rule) is used for $\bar{N}^+$ and $\bar{N}^-$ respectively.
(iv) $\bar{N}^+$ and $\bar{N}^-$ transform in a simple manner under rotations, which is easy to visualise (figure 3).

Note that ν^+ and ν^- are obtained from the y component of the respective companion 'vector' by equation (4.5). They enter explicitly only when the normals $\hat{\lambda}^+$ and $\hat{\lambda}^-$ given by equation (4.2) are determined.

During the projection in (i) above, the $(\hat{x}, \hat{y})$ plane is interpreted as C and, as we have seen, $\bar{N}^-$ gives u_- while $\bar{N}^+$ gives u_+^*. If we regard the plane as C^* $(\hat{x}=1, \hat{y}=-\mathrm{i})$ during the latter projection, we obtain u_+ instead. Alternatively, we may look at the plane from below (along $-\hat{z}$) and interpret it as C, still obtaining u_+. This view is in fact quite natural as seen, for example, when regarding rotations around $\hat{z}$. Then the projections $\bar{F}^+$ and $\bar{F}^-$, which move in circular orbits under the rotation (figure 3), lie in the $(\hat{x}, \hat{y})$ plane and constitute our wanted projections. Both rotate in the positive sense around $\hat{z}$. If we regard the phase of u_- in C (from above) it changes in the positive sense, while regarding u_+ in C (from below) its phase changes in the *negative* sense. Indeed, this is the normal behaviour of standard components. In this manner we may regard the $(\hat{x}, \hat{y})$ plane as a double complex plane, one seen from above and one from below.

Finally, we shall investigate the conjugate spinors. In our double-'vector' visualisation of spinors, the conjugation (3.24) is, in fact, simple to perform. Equations (4.3) and (4.4) yield (observe that $\boldsymbol{B}$ is not changed during a conjugation)

$$u^{\mathrm{c}}=\boldsymbol{B}(\bar{N}^-,\nu^-)\begin{pmatrix}0\\ \mathrm{i}\end{pmatrix},\qquad u^{\mathrm{c}}=\boldsymbol{B}(\bar{N}^+,\nu^+)\begin{pmatrix}-\mathrm{i}\\ 0\end{pmatrix}=\boldsymbol{B}(-\bar{N}^+,-\nu^+)\begin{pmatrix}\mathrm{i}\\ 0\end{pmatrix}.$$

Thus the conjugation simply corresponds to the operation

$$\begin{pmatrix}\bar{N}^+\\ \bar{N}^-\end{pmatrix}\to\begin{pmatrix}\bar{N}^-\\ -\bar{N}^+\end{pmatrix}$$

on our double-'vector'. Note that the relation (4.6) is satisfied for the new 'vectors' as required.

The triad $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$ associated with u is obtained as usual from $(\hat{x}, \hat{y}, \hat{z})$ by the rotation $(\hat{n}^-, \varphi^-)$. After conjugation, this rotation plays the role of $(\hat{n}^+, \varphi^+)$, i.e. $(\hat{\xi}, \hat{\eta}, \hat{\zeta})^{\mathrm{c}}$, the triad associated with u^{c}, is obtained from $(-\hat{x}, \hat{y}, -\hat{z})$ by the rotation $(\hat{n}^-, \varphi^-)$. Thus $(\hat{\xi}, \hat{\eta}, \hat{\zeta})^{\mathrm{c}}=(-\hat{\xi}, \hat{\eta}, -\hat{\zeta})$ which can be written

$$(\hat{\xi}, \hat{\eta}, \hat{\zeta})^{\mathrm{c}}=\boldsymbol{A}_{\hat{\eta}}(\pi)(\hat{\xi}, \hat{\eta}, \hat{\zeta}).$$

This is the analogue of equation (3.27).

5. Hellsten's spinor visualisation model

In this section we shall give the relationship between our way of regarding the spinors and that of Hellsten (1979). We defined a spinor from a rotation ($\hat{n}$, φ) that would take the reference orientation Ω_r to the orientation Ω associated with the spinor. As is evident from § 2, it is sufficient to regard rotation angles $-2\pi \leq \varphi \leq 2\pi$ (cf equation (2.13)). Let S denote a sphere of unit diameter centred at the origin (0) and take a point of reference (P) on S (figure 7). Under the rotation ($\hat{n}$, φ) a mobile point originally located at P will be moved along a circular arc on S from P to another point Q (provided the rotation axis does not pass through P). The arc is a part of a circle (Γ), that equals the intersection between the sphere S and a plane perpendicular to $\hat{n}$ containing P (figure 7). This directed arc (from P to Q) determines ($\hat{n}$, φ), and hence the spinor ($\bar{N}$, ν) uniquely. Note that the complementary arc (from P to Q in the reversed rotation direction) gives the rotation angle $\varphi - 2\pi$ (or $\varphi + 2\pi$ when $\varphi < 0$) and hence the spinor $-(\bar{N}, \nu)$.

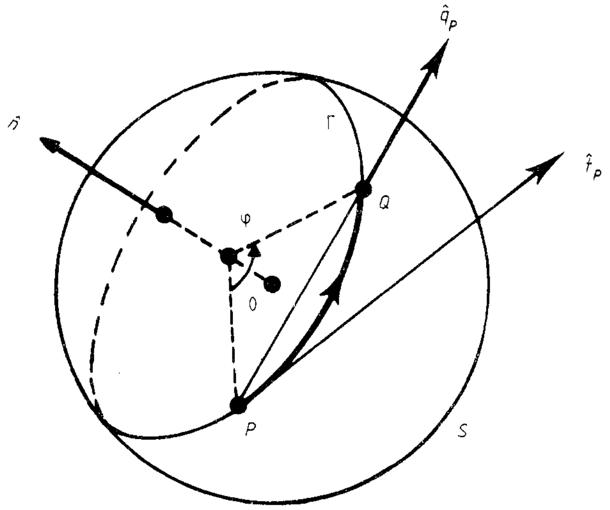

Figure 7. The tangent 'vector' $\hat{t}_P$ and the 'vector' $\hat{q}_P$ used in Hellsten's spinor model define a directed arc PQ of the circle Γ on the sphere S. The arc is also defined from the rotation ($\hat{n}$, φ) of the spinor (see text).

We accordingly have a one-to-one correspondence between spinors and directed arcs with a common starting point (P), unless P and Q coincide (this case will be treated separately later). An arc can also be specified by means of a line through its end points P and Q, and a directed tangent to the arc at P (the direction gives the sense of rotation of Γ). In Hellsten's model a unit 'vector' $\hat{q}_P$ from P through Q (index P denotes that it is drawn from P) and a tangent 'vector' $\hat{t}_P$ at P are used to define the arc (figure 7) and hence the normed spinor. As a matter of fact, these entities are closely related to the spinor in complex notation, as we shall now show.

In order to obtain the precise relationship between our previous results and Hellsten's spinor model, we have to introduce a new convention for our entity ($\bar{N}$, ν).

Take an arbitrary spinor u and define $(\bar{N}^{\uparrow}, \nu^{\uparrow})$ from the relation

$$u = \boldsymbol{B}(\bar{N}^{\uparrow}, \nu^{\uparrow})\begin{pmatrix}0\\1\end{pmatrix} \tag{5.1}$$

(cf equation (4.4)). This means that

$$(\bar{N}^{\uparrow}, \nu^{\uparrow}) = (-\beta_+, -\alpha_+, \beta_-, \alpha_-). \tag{5.2}$$

Since the spinor $\begin{pmatrix}0\\1\end{pmatrix}$ has the associated triad $(\hat{x}, -\hat{y}, -\hat{z})$, it follows that the rotation defined by $(\bar{N}^{\uparrow}, \nu^{\uparrow}) = (\hat{n}^{\uparrow} \sin(\varphi^{\uparrow}/2), \cos(\varphi^{\uparrow}/2))$ would bring $(\hat{x}, -\hat{y}, -\hat{z})$ to coincide with $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$, which is the triad associated with u (cf the corresponding discussion regarding $(\bar{N}^-, \nu^-)$ and $(\bar{N}^+, \nu^+)$ in § 4). Let us choose the reference point P to be the end point of $-\frac{1}{2}\hat{z}$ (the 'south pole' of S). Under the rotation $(\hat{n}^{\uparrow}, \varphi^{\uparrow})$, where $-2\pi \leqslant \varphi^{\uparrow} \leqslant 2\pi$, the vector $-\frac{1}{2}\hat{z}$ would be brought to $\frac{1}{2}\hat{\zeta}$ along an arc on S, which by the definition above is the arc associated with the spinor u. Thus, Q equals the end point of $\frac{1}{2}\hat{\zeta} = \bar{v}^{\mathrm{I}}$ (equation (3.21)).

The tangent $\hat{t}_P$ of the arc is obviously perpendicular to $\hat{n}^{\uparrow}$ and hence to $\bar{N}^{\uparrow}$, which are normals of the plane containing the arc. From equation (5.2) it follows that the projection of $\bar{N}^{\uparrow}$ on the (x, y) plane forms the angle $-\gamma_+$ to the negative y-axis (as before $\alpha_+ + \mathrm{i}\beta_+ = \rho_+ \exp(\mathrm{i}\gamma_+)$). The same is true if we project it on the parallel tangent plane at P, which is spanned by $\hat{x}_P$ and $\hat{y}_P$—the basis vectors $\hat{x}$ and $\hat{y}$ parallelly displaced along the z axis to P. Thus the angle between $\hat{t}_P$ and the x_P axis also equals $-\gamma_+$ (figure 8), and if the tangent plane is interpreted as C^* (or C seen from below) we have

$$\hat{t}_P = \exp(\mathrm{i}\gamma_+). \tag{5.3}$$

If we write the spinor

$$u = \begin{pmatrix}\rho_+ \mathrm{e}^{\mathrm{i}\gamma_+}\\ \rho_- \mathrm{e}^{\mathrm{i}\gamma_-}\end{pmatrix} = \mathrm{e}^{\mathrm{i}\gamma_+}\begin{pmatrix}\rho_+\\ \rho_-\,\mathrm{e}^{\mathrm{i}(\gamma_- - \gamma_+)}\end{pmatrix} \tag{5.4}$$

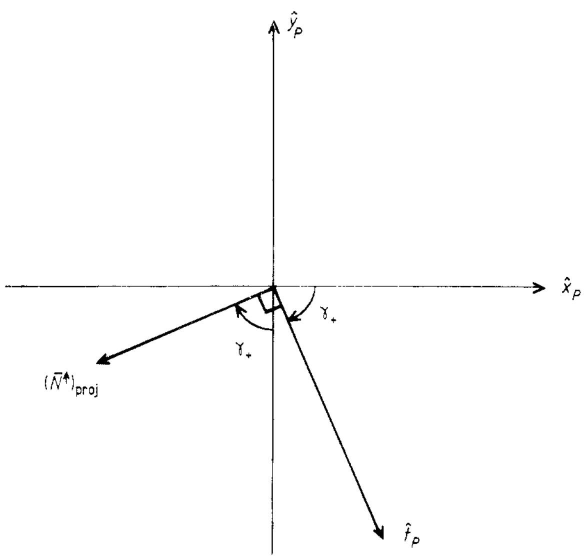

Figure 8. Illustration of the relation between $\bar{N}^{\uparrow}$ and $\hat{t}_P$.

the 'vector' $\hat{t}_P$ accordingly gives its phase (we here assume that $\rho_+ \neq 0$ which means that $Q \neq P$). The remaining part of u is determined by the end point Q of the arc or, equivalently, by $\hat{q}_P$.

The vector OQ equals $\frac{1}{2}\hat{\xi} = \bar{v}^{\mathrm{I}}$ given explicitly by equation (3.23). A short calculation shows that the spherical polar coordinates (r, ϕ, θ) of Q satisfy

$$r = \tfrac{1}{2}, \qquad \phi = \gamma_- - \gamma_+, \qquad \cos\tfrac{1}{2}\theta = \rho_+, \qquad \sin\tfrac{1}{2}\theta = \rho_-. \tag{5.5}$$

In the $(\hat{x}_P, \hat{y}_P, \hat{z}_P)$ coordinate system, Q has the polar angles $(\phi, \theta/2)_P$ (see figure 9). The unit vector $\hat{q}_P$, which passes through Q, obviously has the projections of lengths ρ_+ and ρ_- on the $\hat{z}_P$ axis and the $(\hat{x}_P, \hat{y}_P)$ plane respectively. Hence, we have

$$\hat{q}_P = \begin{pmatrix} \rho_- \cos(\gamma_- - \gamma_+) \\ \rho_- \sin(\gamma_- - \gamma_+) \\ \rho_+ \end{pmatrix}_P . \tag{5.6}$$

If this time we interpret the $(\hat{x}_P, \hat{y}_P)$ plane as C (from above), we obtain exactly the quantities in the right-hand parentheses of equation (5.4).

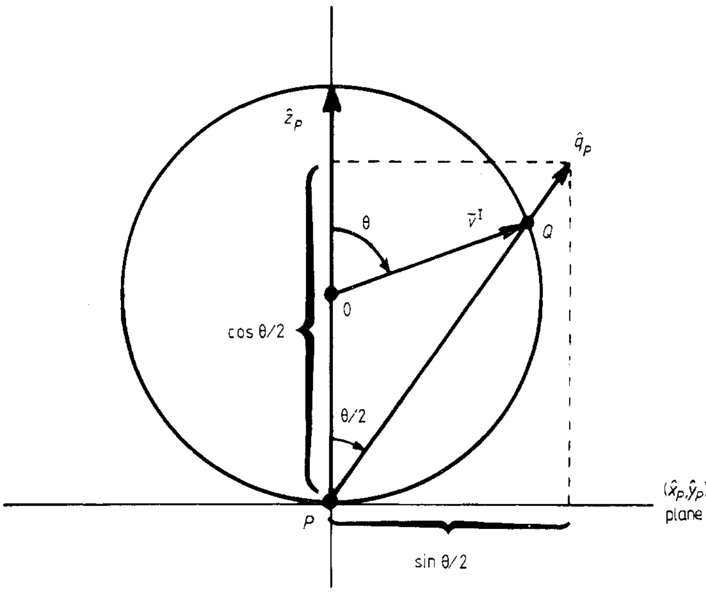

Figure 9. The relationship between $\hat{q}_P$ and the spin eigendirection $\bar{v}^{\mathrm{I}}$ of a spinor. The projections of $\hat{q}_P: \rho_+ = \cos\theta/2$ and $\rho_- = \sin\theta/2$ equal the amplitudes of the spinor components.

Thus, $\hat{t}_P$ and $\hat{q}_P$ together give the spinor directly. If the spinor is not normed, we simply multiply $\hat{q}_P$ by the spinor norm. This is the spinor visualisation of Hellsten (derived differently by him). In general the 'vectors' $\bar{q}_P$ are not additive, but their sum can be constructed by using the parallelogram rule combined with a transformation (Hellsten 1979).

Another way to visualise the phase angle—also used by Hellsten—is to follow the transformation of $\hat{x}_P$ during the rotation given by $(\bar{N}^{\uparrow}, \nu^{\uparrow})$, see figure 10. The transformed vector is $\hat{\xi}_Q$ ($\hat{\xi}$ parallelly displaced along $\bar{v}^{\mathrm{I}}$ to Q). $\hat{\xi}_Q$ is a tangent vector to the

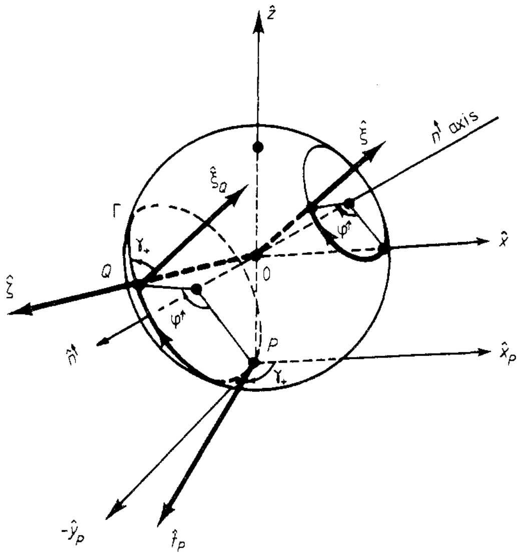

Figure 10. An illustration of the relation between Hellsten's spinor model and the triad formulation (see text).

sphere S. The angle between $\hat{\xi}_Q$ and the circle Γ obviously equals that between $\hat{x}_P$ and Γ (the angle is constant during the rotation) and hence is equal to γ_+ (as shown in figure 10). Thus Q and the tangent vector $\hat{\xi}_Q$—or equivalently the vectors $\hat{\zeta}$ and $\hat{\xi}$—are related geometrically in a fairly simple manner to the normed spinor. However, $\hat{\xi}$ and $\hat{\zeta}$ do not give the sign of the spinor since they leave the rotation direction of Γ undefined. Note that both in this model and in our previous method of visualising a spinor (§ 4) it is necessary to determine the rotations that bring certain triads to coincide with $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$.

It remains to consider the case $\rho_+ = 0$. Then we have $\hat{\zeta} = -\hat{z}$ and $Q = P$. Hence $\hat{\xi}$ lies in the $(\hat{x}, \hat{y})$ plane and the angle between $\hat{\xi}$ and $\hat{x}$ is $\varphi^\uparrow$. From equations (5.1) and (2.17) it follows that $u_- = \exp(\mathrm{i}\varphi^\uparrow/2)$, and equation (5.4) shows that γ_+ is undefined. Let us first choose $\gamma_+ = 0$. Then we have (equation (5.6))

$$\hat{q}_P = \begin{pmatrix} \cos\frac{1}{2}\varphi^\uparrow \\ \sin\frac{1}{2}\varphi^\uparrow \\ 0 \end{pmatrix}$$

which means that $\hat{q}_P$ lies on the line that bisects the angle $\varphi^\uparrow$ between $\hat{\xi}_Q (Q = P)$ and $\hat{x}_P$. The direction of $\hat{q}_P$ on this line, i.e. the sign of the spinor, is not determined by $\hat{\xi}$. If we take an arbitrary γ_+ instead of $\gamma_+ = 0$, the only thing that can be said is that the angle between $\hat{q}_P$ and $\hat{t}_P$ (equation (5.3)) equals $\varphi^\uparrow/2$. However, this is sufficient in order to specify the spinor from $\hat{q}_P$ and $\hat{t}_P$.

Finally, we turn to the transformation of an arbitrary spinor under a rotation $(\hat{m}, \psi)$. The transformation of $\hat{q}_P$ is very simple to determine since $\bar{v}^{\mathrm{I}}$, which defines Q, is a vector and Q is common for both entities. During the rotation Q' moves in a circle that is the intersection of S and a plane perpendicular to $\hat{m}$. The 'vector' $\hat{q}'_P$ crosses S at Q' and has constant length. The change in the remaining phase factor $\exp(\mathrm{i}\gamma_+)$ can be found by means of a geometrical construction, derived by Hellsten (1979).

Acknowledgments

I would like to thank Hans Hellsten for stimulating discussions and Thomas Schuler for checking the English.

Appendix

Here we shall determine the projection of $\bar{G}$ (defined in § 2, see figure 3) on the $\hat{m}$ axis:

$$\nu = \bar{G} \cdot \hat{m}. \tag{A1}$$

$\bar{G}$ lies in the plane Λ which is spanned by $\hat{n}$ and $\bar{h}$ (see figure 2). We can write

$$\bar{G} = c_1\hat{n} + c_2\bar{h} \tag{A2}$$

where c_1 and c_2 are coefficients to be determined. Since $\bar{G}$ is orthogonal to $\bar{f}$, we can obtain a relation between c_1 and c_2 by multiplying equation (A2) with $\bar{f}$. Using this relation and the facts (from equation (2.2)) that $\hat{n} \cdot \bar{f} = 1 - (\hat{n} \cdot \hat{m})^2 = |\bar{f}|^2$ and $\bar{h} \cdot \bar{f} = -(\hat{n} \cdot \hat{m})(\bar{h} \cdot \hat{m})$, we obtain from equations (A1)–(A2)

$$\nu = c_2(\bar{h} \cdot \hat{m}/|\bar{f}|^2). \tag{A3}$$

Furthermore (cf equation (2.7) and figure 3), $\bar{F} = \bar{E} \times \hat{m} = \bar{G} \times \hat{m}$. Applying this to equation (A2), we can compute $\bar{F} \cdot \bar{f} = (\bar{G} \times \hat{m}) \cdot \bar{f}$. This expression can be simplified if we use the relations (from equation (2.2)) $(\hat{n} \times \hat{m}) \cdot \bar{f} = 0$ and $(\bar{h} \times \hat{m}) \cdot \bar{f} = (\bar{h} \times \bar{k}) \cdot \hat{n} = |\bar{h}||\bar{F}|$, where the last equality is a consequence of equations (2.3)–(2.4) and $\bar{F} = \bar{f} \sin(\varphi/2)$. The result is

$$c_2 = |\bar{f}|/|\bar{h}|. \tag{A4}$$

Equation (2.2) yields $\bar{h} \cdot \hat{m} = \bar{h} \cdot \bar{k}$, and accordingly we obtain from equations (A3)–(A4) and (2.3)

$$\nu = \bar{h} \cdot \bar{k}/|\bar{h}||\bar{k}| = \cos \tfrac{1}{2}\varphi, \tag{A5}$$

because the angle between $\bar{h}$ and $\bar{k}$ equals $\varphi/2$.

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